Problem Set 6 (Due 2/26/21 11:59PM)

Your Name:

Names of Any Collaborators:

Please review the problem set guidelines in the syllabus before turning in your work.

P1. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists $M > 0$ and $N \in \mathbb{N}$ such that

$$|f(z)| \leq M(1 + |z|^N)$$

for all $z \in \mathbb{C}$. Prove that f is a polynomial of degree at most N.

Proof. Since f is entire, it has a Taylor series expansion about 0 that converges everywhere in the complex plane:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n. \tag{$*$}$$

We prove that $f^{(n)}(0) = 0$ for all $n > N$. Let $R > 0$. Then since f is entire, it is analytic everywhere inside and on the circle $C_R(0)$. We apply the Cauchy Inequality with $n > N$:

$$\begin{aligned} |f^{(n)}(0)| &\leq \frac{n!}{R^n} \max_{z \in C_R(0)} |f(z)| \\ &\leq \frac{n!}{R^n} \max_{z \in C_R(0)} M(1 + |z|^N) \\ &= \frac{n!}{R^n} M(1 + R^N). \end{aligned}$$

But $n > N$ so the right hand side converges to 0 as $R \to \infty$. This proves that $f^{(n)}(0) = 0$ for all $n > N$. Hence, using ($*$),

$$f(z) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n = a_0 + \frac{f^{(1)}(0)}{1!} z + \frac{f^{(2)}(0)}{2!} z^2 + \cdots + \frac{f^{(N)}(0)}{N!} z^N$$

which is a polynomial of at most degree N. This proves the claim. ☕

P2. (a) Find the Taylor series expansion for $f(w) = \frac{1}{w}$ on the disk $D_1(1) = \{w \in \mathbb{C} : |w - 1| < 1\}$.

(b) Let $\operatorname{Log} z$ be the principal branch of the logarithm. Let $z \in D_1(1)$ and let C be a contour lying interior to $D_1(1)$ and joining $w = 1$ to $w = z$. Find the Taylor series expansion for $\operatorname{Log} z$ on the disk $D_1(1)$ by integrating the Taylor series expansion found in part (a) over the contour C.

Solution. (a) We seek a Taylor series centered at $z_0 = 1$. The condition $|1-w| = |w-1| < 1$ suggests we use the geometric series. We have

$$\begin{aligned}\frac{1}{w} &= \frac{1}{1-(1-w)} \\ &= \sum_{n=0}^{\infty}(1-w)^n && \text{(since } |1-w|<1) \\ &= \sum_{n=0}^{\infty}(-1)^n(w-1)^n.\end{aligned}$$

Since this is a power series centered at $z_0 = 1$ and converging to $\frac{1}{w}$ on $D_1(1)$, it must in fact be the Taylor series of $\frac{1}{w}$ on that disk.

(b) Following the suggestion, let $z \in D_1(1)$ and let C be any contour lying interior to $D_1(1)$ and joining $w = 1$ to $w = z$. Notice that the principal branch $\operatorname{Log} z$ is defined everywhere on $D_1(1)$ since $D_1(1)$ contains no point of the branch cut (the ray $\theta = \pi$). Hence, $\operatorname{Log} w$ is analytic on $D_1(1)$ and thus $\frac{d}{dw}\operatorname{Log} w = \frac{1}{w}$. That is, $\operatorname{Log} w$ is an antiderivative of $\frac{1}{w}$ on the disk. Now, we integrate both sides of the Taylor series found in (a). We have

$$\begin{aligned}\operatorname{Log} z &= \int_C \frac{1}{w}\,dw && \text{(Fund. Thm. of Contour Integrals)} \\ &= \int_C \sum_{n=0}^{\infty}(-1)^n(w-1)^n\,dw \\ &= \sum_{n=0}^{\infty}(-1)^n\int_C (w-1)^n\,dw && \text{(Thm. on Integrating Power Series)} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}(z-1)^{n+1}\,dz && \text{(Fund. Thm. of Contour Integrals)} \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n\,dz.\end{aligned}$$

Since this is a power series centered at $z_0 = 1$ and converging to $\operatorname{Log} z$ on $D_1(1)$, it must be the Taylor series of $\operatorname{Log} z$ on that disk.

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P3. For each function, compute the Laurent series on the given annulus.

(a) $f(z) = z^3 \sin\left(\frac{1}{z^3}\right)$, $0 < |z| < \infty$.

(b) $f(z) = \frac{1}{2-z}$, $2 < |z| < \infty$.

(c) $\frac{z}{1+z^2}$, $0 < |z-i| < 2$.

(d) $f(z) = \frac{z}{(z-1)(z-3)}$, $0 < |z-1| < 2$.

(e) $f(z) = \frac{z}{(z-1)(z-3)}$, $2 < |z-1| < \infty$.

Solution. (a) The MacLaurin series $\sin w = \sum_{n=0}^{\infty}\frac{(-1)^n w^{2n+1}}{(2n+1)!}$ is valid everywhere on $\mathbb{C}$. In particular, it is valued for $w = \frac{1}{z}$ where $0 < |z| < \infty$. Hence, we have

$$\begin{aligned} z^3 \sin\left(\frac{1}{z^3}\right) &= z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{6n+3}(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{6n}(2n+1)!}. \end{aligned}$$

(b) For any $z \in \mathbb{C}$ satisfying $2 < |z| < \infty$, we have $\left|\frac{2}{z}\right| < 1$. Hence,

$$\begin{aligned} \frac{1}{2-z} &= -\frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} \\ &= -\frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} && \text{(since } 2 < |z| < \infty\text{)} \\ &= -\sum_{n=1}^{\infty} \frac{2^{n-1}}{z^n}. \end{aligned}$$

(c) We seek a Laurent series involving powers of $z - i$. We will try to use the condition $\left|\frac{z-i}{2}\right| < 1$. We will begin by using the partial fraction decomposition. We have

$$\begin{aligned} \frac{z}{1+z^2} &= \frac{1}{2}\left(\frac{1}{z+i} + \frac{1}{z-i}\right) \\ &= \frac{1}{2}\left(\frac{1}{2i} \cdot \frac{1}{1-\frac{i-z}{2i}} + \frac{1}{z-i}\right) \\ &= \frac{1}{2}\left(\frac{1}{2i} \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(2i)^n} + \frac{1}{z-i}\right) && \left(\text{since } \left|\frac{i-z}{2i}\right| < 1\right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(2i)^{n+1}} + \frac{1}{2} \cdot \frac{1}{z-i}. \end{aligned}$$

(d) We seek a Laurent series involving powers of $z - 1$. We will try to use the condition $\left|\frac{z-1}{2}\right| < 1$. We begin by using the partial fraction decomposition. We have

$$\begin{aligned} \frac{z}{(z-1)(z-3)} &= \frac{3}{2} \cdot \frac{1}{z-3} - \frac{1}{2} \cdot \frac{1}{z-1} \\ &= -\frac{3}{4} \cdot \frac{1}{1-\frac{z-1}{2}} - \frac{1}{2} \cdot \frac{1}{z-1} \\ &= -\frac{3}{4} \cdot \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} - \frac{1}{2} \cdot \frac{1}{z-1} && \left(\text{since } \left|\frac{z-1}{2}\right| < 1\right) \\ &= -3 \cdot \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2} \cdot \frac{1}{z-1}. \end{aligned}$$

(e) We seek a Laurent series involving powers of $z - 1$. We will try to use the condition

$\left|\frac{2}{z-1}\right| < 1$. We begin by using the partial fraction decomposition. We have

$$\begin{aligned}\frac{z}{(z-1)(z-3)} &= \frac{3}{2}\cdot\frac{1}{z-3} - \frac{1}{2}\cdot\frac{1}{z-1} \\ &= \frac{3}{2(z-1)}\cdot\frac{1}{1-\frac{2}{z-1}} - \frac{1}{2}\cdot\frac{1}{z-1} \\ &= \frac{3}{2(z-1)}\sum_{n=0}^{\infty}\frac{2^n}{(z-1)^n} - \frac{1}{2}\cdot\frac{1}{z-1} \\ &= 3\sum_{n=0}^{\infty}\frac{2^{n-1}}{(z-1)^{n+1}} - \frac{1}{2}\cdot\frac{1}{z-1} \\ &= \frac{3}{2}\cdot\frac{1}{z-1} + 3\sum_{n=1}^{\infty}\frac{2^{n-1}}{(z-1)^{n+1}} - \frac{1}{2}\cdot\frac{1}{z-1} \\ &= \frac{1}{z-1} + 3\sum_{n=2}^{\infty}\frac{2^{n-2}}{(z-1)^n}.\end{aligned}$$

This completes the solution. ☕

P4. Prove that the function

$$f(z) = \begin{cases}\frac{1-\cos z}{z^2}, & z \neq 0 \\ \frac{1}{2}, & z = 0\end{cases}$$

is entire.

Proof. Since $\cos z$ is entire, by Taylor's theorem we can write

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}$$

for any $z \in \mathbb{C}$. Hence, whenever $z \neq 0$, we have

$$\frac{1-\cos z}{z^2} = \frac{1}{z^2}\left(1 - \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\right) = \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n-2}}{(2n)!}.$$

But, the first term of this series is $\frac{1}{2}$ (when $n=1$) and every other term involves a power of z. Hence, when $z = 0$, we have

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n-2}}{(2n)!}.$$

Thus, we have proved that for all $z \in \mathbb{C}$, $f(z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n-2}}{(2n)!}$. But the right-hand side of this equation is a power series with disk of convergence equal to the whole complex plane. Since a power series represents an analytic function on its disk of convergence, we can conclude that f is entire. ☕

P5. (Optional)[1] Use complex analysis to prove the formula:[2]

$$\sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{7^n} = \sqrt{\frac{7}{3}}$$

[1]Worth 4 points of extra credit, added to you score on this assignment.
[2]Hint: begin by applying the formula for $\binom{2n}{n}$ from P5 on PSet4.

(This problem is fantastic, and surprisingly relevant.)

Proof. Let $n \in \mathbb{N}$ and let C be the unit circle with positive orientation. Consider the series $\sum_{n=0}^{\infty} w^n$ with $w = \frac{(1+z)^2}{7z}$. This is geometric series. Given any $z \in C$, we have $|z| = 1$ and hence

$$|w| = \left|\frac{(1+z)^2}{7z}\right| = \frac{|1+z|^2}{7\,|z|} \leq \frac{(1+|z|)^2}{7\,|z|} = \frac{4}{7}.$$

This proves the the series converges uniformly on C. Now, let $g(z) = \frac{1}{z}$. It is clear that $g(z)$ is continuous on C and lies in the disk of convergence of the series. Hence, the theorem on integrating power series applies and we can write

$$\int_C \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{(1+z)^2}{7z}\right)^n dz = \sum_{n=0}^{\infty} \int_C \frac{1}{z} \left(\frac{(1+z)^2}{7z}\right)^n. \tag{$*$}$$

Now, we will compute the series of real numbers. According to P5 from PSet 4, we can write

$$\binom{2n}{n} = \frac{1}{2\pi i} \int_C \frac{(1+z)^{2n}}{z^{n+1}} dz. \tag{$**$}$$

Then we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{7^n} &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{7^n} \int_C \frac{(1+z)^{2n}}{z^{n+1}} dz && \text{(by $(**)$)} \\
&= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_C \frac{1}{z} \left(\frac{(1+z)^2}{7z}\right)^n \\
&= \frac{1}{2\pi i} \int_C \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{(1+z)^2}{7z}\right)^n dz && \text{(by $(*)$)} \\
&= \frac{1}{2\pi i} \int_C \frac{1}{z} \left(\frac{1}{1 - \frac{(1+z)^2}{7z}}\right) dz && \text{(Sum of geometric series)} \\
&= \frac{7}{2\pi i} \int_C \frac{1}{7z - (1+z)^2} dz \\
&= \frac{7}{2\pi i} \int_C \frac{1}{5z - 1 - z^2} dz.
\end{aligned}$$

Finally, to compute $\int_C \frac{1}{5z-1-z^2}\, dz$, observe that the roots of $5z - 1 - z^2$ are $z^+ = \frac{5+\sqrt{21}}{2}$ and $z^- = \frac{5-\sqrt{21}}{2}$. Hence, we can write $\frac{1}{5z-1-z^2} = \frac{\frac{1}{z-z^+}}{z-z^-}$. Moreover, z^- lies interior to C and z^+ lies exterior to C. This implies that $\frac{1}{z-z^+}$ is analytic everywhere on and interior to C and thus, by the Cauchy Integral Formula, we obtain

$$\int_C \frac{1}{5z-1-z^2}\, dz = \int_C \frac{\frac{1}{z-z^+}}{z-z^-}\, dz = \frac{2\pi i}{z^- - z^+} = \frac{2\pi i}{\sqrt{21}}.$$

At last, we conclude that

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{7^n} = \frac{7}{2\pi i} \int_C \frac{1}{5z-1-z^2}\, dz = \frac{7}{\sqrt{21}} = \sqrt{\frac{7}{3}}.$$

That was wild! ☕